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AN EVALUATION OF SOME APPROXIMATE METHODS OF COMPUTING LANDING STRESSES IN AIRCRAFT

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SUMMARY

An investigation is made to obtain some information concerning the nature and magnitude of the errors involved in computing the landing stresses for large and relatively flexible aircraft when several simplifying assumptions are made. An exact solution is made for the landing stresses of a simplified structure and is compared with several approximate solutions made when the simplifying assumptions are used.

The simplified structure investigated consisted of a uniform beam for the wing, a concentrated mass for the fuselage, and an undamped linear spring for the landing gear. This structure was considered to be in uniform translation until the landing gear touched the ground. The subsequent motion was computed by using operational calculus in conjunction with standard beam theory. In general, it was found that, for moderately flexible landing gears, the neglect of the effect of structural elasticity in computations of strut forces or of the acceleration of the point of attachment of the landing gear and then the computations of stresses by considering the structure to be elastic led to small conservative errors; whereas the neglect of structural elasticity in computing wing stresses from the strut force or acceleration of the point of attachment of the landing strut led to unconservative errors of appreciable magnitude. This result suggests that a satisfactory treatment of the landing problem may possibly be obtained from an analysis which assumes that in landing the aircraft is an elastic structure subject to the forces or accelerations found in a drop test in which a rigid mass is used.

INTRODUCTION

When an aircraft lands, the vertical component of its velocity is rather suddenly reduced to zero. This sudden change in motion of the aircraft gives rise to stresses within the structure which may become large and even destructive as the size and weight of airplanes increase and the design load factor decreases. The design of large airplanes should therefore consider the effect of a severe landing on the wings, fuselage, tail surface, landing-gear struts, and other elastic parts of the airplane structure.

When the stiffness of an airplane structure is relatively large compared with the stiffness of the landing gear, all parts of the airplane are subject to essentially the same acceleration during landing and the stresses in all parts of the airplane are therefore readily computed by methods of statics. When the stiffnesses of the landing gear and the airplane structure are comparable in magnitude, the various parts of the airplane have different accelerations and the problem becomes much more involved. The calculation of the stresses is a complex problem even when the equations involved are purely linear; that is, when the internal forces are proportional to the deformations. It is much more complicated in the actual aircraft because of the nonlinear characteristics of the landing gear. A number of approximate methods of computing landing stresses have consequently been used.

Because of the practical importance of the landing problem, it is of some interest to determine the nature and magnitude of the error involved in various approximate methods that have been used. For this purpose, an exact solution is made to determine the landing stresses for a highly simplified structure in which a uniform beam is used to represent the wing, a rigid mass the fuselage, and a simple spring the landing gear.

The stresses in the wings excited by the landing impact are computed by operational calculus in conjunction with the standard engineering vibration theory of beams. The results are compared with the results found by a number of approximate methods. The analytical treatment of the exact and approximate solutions are given in appendixes.

SYMBOLS

E	modulus of elasticity
γ	density of wing material in units of weight
c	velocity of sound in wing material $\left(\sqrt{\frac{Eg}{\gamma}}\right)$
g	acceleration of gravity
L	semispan of wing
I	moment of inertia of cross section of wing about neutral axis
A	cross-sectional area of wing
ρ	radius of gyration of cross section of wing $\left(\sqrt{\frac{I}{A}}\right)$
x	coordinate along wing measured from root
y	distance from neutral axis of wing to any fiber

m	mass of wing (semispan)
M	concentrated mass (one-half of fuselage mass)
S	spring stiffness
t	time, zero at beginning of impact
p	operator $\left(\frac{\partial}{\partial t}\right)$
n	integers 1, 2, 3, and so forth designating a particular mode of vibration
θ_n	nth positive root of transcendental equation associated with a given type of vibration
ω_n	undamped natural angular frequency of nth mode, radians per second
v	vertical velocity of aircraft prior to impact
ω_c	natural frequency of fundamental mode of a cantilever, radians per second $\left(\rho c \frac{3.52}{L^2}\right)$
ω_0	natural frequency of fuselage-spring system, radians per second $\left(\sqrt{\frac{S}{M}}\right)$
ω_s	natural frequency of airplane with wing rigid, radians per second $\left(\sqrt{\frac{S}{M + m}}\right)$
$w(x, t)$	deflection, relative to root position at $t = 0$, of wing at station x and time t
$a(x, t)$	acceleration of wing at station x and time t
$\sigma(x, y, t)$	bending stress in wing at station x, distance from neutral axis y, and time t
$\bar{\tau}(x, t)$	average shear stress over cross section of beam at station x and time t
A_n	bending-stress coefficient
$\bar{A}$	maximum bending-stress coefficient obtained from first three modes with proper regard to phase

B_n shear-stress coefficient

B maximum shear-stress coefficient obtained from first three modes with proper regard to phase

RESULTS AND DISCUSSION

Exact solution.- In order to obtain pertinent information on the problem of landing impacts, an exact solution is made of the landing stresses of a highly simplified structure. In this simplified structure (fig. 1), a uniform beam of mass m was used to represent the wings of the airplane, a rigid mass of magnitude M to represent the fuselage, and a simple spring of stiffness S was substituted for the landing gear. The exact analytical treatment giving the equations for frequencies, deflections, accelerations, strut force, bending stresses, and shear stresses is presented in appendix A. The maximum root bending stress that results from impact (gravity not included) is shown in this appendix to be given by the equation

$$\sigma = \bar{A} \frac{V}{c} \frac{y}{\rho} E$$

where $\bar{A}$ is a dimensionless coefficient dependent on the physical parameters of the structure. As can be seen, the maximum stress is directly proportional to the velocity of descent.

In figure 2 the coefficient $\bar{A}$ is given for several values of the ratio of fuselage mass to wing mass as a function of the ratio ω_s/ω_c. In this ratio, ω_c is the fundamental frequency of the wing as a cantilever and ω_s is the frequency of the airplane when the wing is considered rigid. Low values of the frequency ratio correspond to a flexible landing gear and the corresponding induced stresses are relatively small but become larger as the landing gear becomes stiffer. The limiting case of a rigid landing gear $\left(\frac{\omega_s}{\omega_c} = \infty\right)$ was investigated in reference 1, damping being taken into account. The results showed that damping eliminates the higher frequencies much faster than the lower ones so that only the lower modes might be expected to contribute to the maximum root bending stress. In the computation of the curves shown in figure 2, only the first three modes were considered, with proper regard being given to phase. (See appendix A for the stress that is associated with each mode.) On this basis $\bar{A}$ is approximately equal to 2.8 in the case of a rigid landing gear.

Approximate solutions.- The exact solution just discussed was obtained by solving the equations of motion directly. In the approximate solutions, the problem is broken arbitrarily into two parts or stages as follows:

Stage 1: Determination of the strut reaction or of the acceleration of the points of attachment of the landing gear.

Stage 2: Computation of the stresses in the airplane structure by use of one of the quantities obtained in stage 1.

In both stages of this approach the structural elasticity must be properly taken into account if the correct solution is to be obtained. In the approximate solutions the effects of the structural elasticity are neglected in one or both stages. Five approximate solutions are given in appendix B. For convenience in discussion, the approximate methods are identified herein as follows:

Method A - Structural elasticity neglected in both stage 1 and stage 2.

Method B - Structural elasticity considered in determining reaction (stage 1) but neglected in stage 2.

Method C - Structural elasticity considered in determining accelerations (stage 1) but neglected in stage 2.

Method D - Structural elasticity neglected in determining reaction (stage 1) but considered in stage 2.

Method E - Structural elasticity neglected in determining accelerations (stage 1) but considered in stage 2.

Method F - Statistical approach of Biot and Bisplinghoff (reference 2); structural elasticity considered in stage 2.

The simplest calculation, of course, results from use of Method A, which neglects the structural elasticity altogether. The acceleration of all parts of the structure is then assumed to be equal to the acceleration measured in a drop test in which a rigid mass equal to the mass of the airplane without the landing gear is used. The stresses are obtained by statics, from a wing loading obtained by multiplying the mass distribution by the acceleration found in the drop test. The coefficients for maximum root bending stress obtained by method A are shown in figure 3. This curve is independent of the mass ratio M/m. For comparison the exact solution for $\frac{M}{m} = 2$ is also shown. This mass ratio is used for all of the succeeding comparisons.

The physical assumptions of the two approximate methods in which the structural elasticity is taken into account in stage 1 but ignored in stage 2 are shown schematically in figure 4. The results found by method B, where the true strut reaction (strut force given by exact solution) is applied to a rigid airframe, and by method C, where the rigid airframe is subject to the true acceleration (acceleration given by exact solution) at the points of support are shown in figure 3.

It appears from figure 3 that the three approximate methods, which neglect the effects of elasticity in the second stage, are unconservative for $\frac{\omega_s}{\omega_c} < 1$ $\left(\text{markedly so near } \frac{\omega_s}{\omega_c} = 0.5\right)$ but approach the correct results as the ratio approaches zero. For aircraft with extremely stiff landing gears $\left(\frac{\omega_s}{\omega_c} \gg 1\right)$, the approximate methods are highly conservative; they predict infinite stresses when a rigid landing gear is used. The fact that curves obtained by methods B and C, which represent approximate methods in which structural elasticity is taken into account in the first stage but neglected in the second, agree much better with the curve obtained by method A, which neglects structural elasticity altogether, than with the exact solution suggests that the neglect of structural elasticity in stage 2 is much more serious than in stage 1.

Two methods are then resorted to in which the effects of structural elasticity are neglected in stage 1 of the analysis but are properly taken into account in stage 2 (methods D and E). The physical assumptions made are indicated schematically in figure 5. In these methods the strut reaction and the acceleration are determined in a drop test in which a rigid mass is used. The resulting stresses are then computed with due regard for the elastic response of the structure. The results found are compared with the exact solution in figure 6. The curve shown for method D is for the strut-reaction method and the curve for method E is for the acceleration method. The curves are cut off when ω_s/ω_c is about 0.7, since for higher ratios the force or accelerations obtained from a drop test with the simple undamped spring will give rise to resonance effects having very little relation to the actual landing problem. It appears from figure 6 that the two methods which neglect structural elasticity only in stage 1 (that is when determining strut forces or the accelerations of the points of support) are conservative and are subject to only small errors.

A somewhat different method of handling the landing problem is the statistical approach developed by Biot and Bisplinghoff in reference 2. In this method (method F), the time history of the landing impact is assumed to be independent of the elastic properties of the structure, so that it may perhaps be classed with the methods which neglect structural elasticity in stage 1. A number of other approximations are also involved for the sake of simplifying the analysis and including a wide variety of landing conditions. Among these approximations are:

(a) The impact force is characterized by only two parameters, the maximum value and the duration of the force; thus, the detailed history of the collision is not considered.

(b) The maximum stress in the first mode is obtained by assuming, in effect, that of a whole class of typical force-time histories having prescribed values for these two parameters; the force-time history that applies is the one which leads to the highest stress.

(c) Similar assumptions are made in obtaining stresses in higher modes. This procedure results, effectively, in the assumption that, of the force-time histories used to determine the envelope response curve, a different one may apply to each mode.

(d) The maximum stress is found by adding together the maximum stresses found for the first three modes without regard to phase.

All of the approximations discussed in connection with the Biot and Bisplinghoff method are conservative except the use of only three modes. The restriction to three modes, which characterizes also the curves for the exact solution of the present paper, would be unconservative in the undamped case; the airplane, however, is subject to a large amount of damping and no real unconservatism is likely to result. The expectation of a conservative result for the Biot and Bisplinghoff method (method F) is verified in figure 6. This method, which makes use of a number of conservative simplifying assumptions, appears in some cases to overestimate the stresses by a factor of almost 2.

CONCLUDING REMARKS

The problem of computing the landing stresses for a large and relatively flexible aircraft is so complex that most investigations are based on simplifying assumptions. The present paper constitutes an attempt to obtain some information concerning the nature and magnitude of the errors in these assumptions by solving the landing problem exactly for a simplified structure and comparing the results with solutions to the same problem obtained by use of the simplifying assumptions.

The simplified structure investigated consisted of a uniform beam for the wing, a concentrated mass for the fuselage, and an undamped linear spring for the landing gear. This structure was considered to be in uniform translation until the landing gear touched the ground. The subsequent motion was computed by using operational calculus in conjunction with standard beam theory.

In most of the approximate treatments that have been proposed the problem is arbitrarily broken into two parts in the first of which the strut reaction or the acceleration of the point of attachment to the landing gear is determined (stage 1), and in the second of which the stresses resulting from the applied force or acceleration are calculated (stage 2). The various approximate methods investigated either neglect altogether or treat only approximately the effects of structural elasticity in one or both stages.

In general, for moderately flexible landing gears, the neglect of the effects of structural elasticity in stage 2 was found to be more serious than the corresponding neglect in stage 1. Such neglect in stage 2 led to unconservative errors of appreciable magnitude; in stage 1 it led to errors which were smaller and on the conservative side. A statistical approach proposed by Biot and Bisplinghoff was found to be always conservative and to have in some cases a safety factor nearly equal to 2.

The conclusions just stated were based primarily on the analysis of the behavior of the simplified structure studied for a ratio of fuselage mass to wing mass equal to 2. The results, however, are essentially the same when the ratio is 1/2 or 5. It therefore appears reasonable to expect that the conclusions of this paper have general validity as applied to conventional aircraft. In addition, the results suggest that a satisfactory treatment of the landing problem may possibly be obtained from an analysis which assumes that, in landing, the aircraft is an elastic structure subject to the forces or accelerations found in a drop test in which a rigid mass is used.

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APPENDIX A

EXACT SOLUTION

General analysis.- In order to make the problem of computing the landing stresses of an airplane susceptible to accurate solution without an inordinate amount of labor it is necessary to idealize the structure. The simplified structure used in the present analysis to represent the airplane is shown in figure 1. The airplane is considered to be falling with a constant velocity v until the bottom of the spring is suddenly stopped by contact with the ground. This disturbance gives rise to oscillations in the beam governed by the differential equation (reference 1)

$$E\rho^2 \frac{\partial^4 w}{\partial x^4} + \frac{\gamma}{g} \frac{\partial^2 w}{\partial t^2} = 0 \tag{A1}$$

Previous analyses (references 1, 3, and 4) have treated special cases of oscillations of a cantilever beam due to impact, with an internal damping term included. Experience indicates, however, that damping has only a slight effect upon the terms that are significant and therefore damping is neglected in the present analysis.

With the use of the notation $c^2 = \frac{Eg}{\gamma}$ and the operational notation $p = \frac{\partial}{\partial t}$, equation (A1) may be written as the ordinary fourth-order differential equation

$$\frac{d^4 w}{dx^4} + \frac{p^2}{c^2 \rho^2} w = 0 \tag{A2}$$

The general solution of this equation is

$$w = P \cosh \theta \frac{x}{L} + Q \sinh \theta \frac{x}{L} + R \sin \theta \frac{x}{L} + S \cos \theta \frac{x}{L} \tag{A3}$$

where

$$\theta = L \sqrt{\frac{ip}{\rho c}}$$

The coefficients P, Q, R, and S are to be evaluated from the boundary conditions, which in this case are

$$\left(\frac{\partial w}{\partial x}\right)_{x=0} = \left(\frac{\partial^2 w}{\partial x^2}\right)_{x=L} = \left(\frac{\partial^3 w}{\partial x^3}\right)_{x=L} = 0$$

$$M\left(\frac{\partial^2 w}{\partial t^2}\right)_{x=0} + EI\left(\frac{\partial^3 w}{\partial x^3}\right)_{x=0} = S\left[\int (v - v\mathit{1})\, dt - w_{(x=0)}\right]$$

The last boundary condition expresses the equilibrium of the forces that act on the mass M. The expression in brackets on the right hand is the change in length of the spring and, when multiplied by S, gives the force exerted by the spring on the mass. The term $\int (v - v\mathit{1})\, dt$ indicates that the motion of uniform translation at the bottom of the spring is suddenly stopped at $t = 0$, and the term $w_{(x=0)}$ is the displacement of the root and is equal to the displacement of the bottom of the spring for $t < 0$. The oscillations set up when the bottom of the spring is suddenly arrested from uniform translation would be the same as if the bottom of the spring were suddenly set in uniform motion with the system initially at rest. The uniform-velocity term may therefore be omitted and the last boundary condition becomes, if $\int (- v\mathit{1})\, dt$ is replaced by the operational form $-\frac{v\mathit{1}}{p}$

$$M\left(\frac{\partial^2 w}{\partial t^2}\right)_{x=0} + EI\left(\frac{\partial^3 w}{\partial x^3}\right)_{x=0} = - S\left[\frac{v\mathit{1}}{p} + w_{(x=0)}\right]$$

With the application of the boundary conditions to equation (A3) there is obtained a set of four nonhomogenous equations in terms of the four coefficients P, Q, R, and S. These equations are solved for the four coefficients and the equation for velocity may then be written. The operational form for the velocity (that induced in the beam when the bottom of the spring is suddenly set in motion) is found to be

$$pw = \frac{-v\mathit{1}}{2} \frac{F\left(\theta \frac{x}{L}\right)}{Z} \qquad \text{(A4)}$$

where

$$F\left(\theta\frac{x}{L}\right) = (1 + \cos\theta\cosh\theta)\left(\cosh\theta\frac{x}{L} + \cos\theta\frac{x}{L}\right)$$

$$+ \sin\theta\sinh\theta\left(\cosh\theta\frac{x}{L} - \cos\theta\frac{x}{L}\right)$$

$$- (\cosh\theta\sin\theta + \sinh\theta\cos\theta)\left(\sinh\theta\frac{x}{L} - \sin\theta\frac{x}{L}\right)$$

$$Z = \frac{p^2}{\omega_0^{\,2}}\left[\left(1 + \frac{\omega_0^{\,2}}{p^2}\right)(1 + \cos\theta\cosh\theta) + \frac{m}{M\theta}(\cosh\theta\sin\theta + \sinh\theta\cos\theta)\right]$$

and

$$\omega_0 = \sqrt{\frac{S}{M}}$$

Interpretation of equation (A4) by the Heaviside expansion theorem and addition of the constant velocity v gives for the total velocity

$$\frac{\partial w(x,\,t)}{\partial t} = v - v\mathit{1} + v\sum_{n=1}^{\infty}\frac{\omega_0^{\,2}}{\omega_n^{\,2}}\,\frac{F\left(\theta_n\frac{x}{L}\right)}{\Delta}\cos\omega_n t\,\mathit{1} \tag{A5}$$

where θ_n is the nth positive root of the equation

$$\left(1 - \frac{\omega_0^{\,2}}{\omega_n^{\,2}}\right) + \frac{m}{M}\,\frac{\cosh\theta_n\sin\theta_n + \sinh\theta_n\cos\theta_n}{\theta_n(1 + \cos\theta_n\cosh\theta_n)} = 0 \tag{A6}$$

and

$$\omega_n = \rho c\,\frac{\theta_n^{\,2}}{L^2}$$

$$F\left(\theta_n\frac{x}{L}\right) = F\left(\theta\frac{x}{L}\right) \qquad \text{(with } \theta \text{ replaced by } \theta_n\text{)}$$

$$\Delta = \left(1 - \frac{\omega_0^2}{\omega_n^2}\right)\frac{\theta_n}{2}(\sinh\theta_n \cos\theta_n - \cosh\theta_n \sin\theta_n)$$

$$+ 2\left(1 + \cos\theta_n \cosh\theta_n\right) + \frac{m}{M}\left[\cos\theta_n \cosh\theta_n\right.$$

$$\left. + \frac{3}{2\theta_n}\left(\cosh\theta_n \sin\theta_n + \sinh\theta_n \cos\theta_n\right)\right]$$

$$= \frac{1}{2\left(1 + \cos\theta_n \cosh\theta_n\right)}\left[1 + 3\frac{\omega_0^2}{\omega_n^2}\left(1 + \cos\theta_n \cosh\theta_n\right)^2\right.$$

$$\left. + \frac{m}{M}\left(\cos\theta_n + \cosh\theta_n\right)^2\right]$$

The term $\frac{\omega_0^2}{\omega_n^2}$ may be transformed into the form $\frac{SL^3}{EI}\frac{m}{M}\frac{1}{\theta_n^4}$ and thus equation (A6), which defines the root θ_n, may be written

$$\left(1 - \frac{SL^3}{EI}\frac{m}{M}\frac{1}{\theta_n^4}\right) + \frac{m}{M}\frac{\cosh\theta_n \sin\theta_n + \sinh\theta_n \cos\theta_n}{\theta_n(1 + \cos\theta_n \cosh\theta_n)} = 0$$

Solution of this equation for SL^3/EI gives

$$\frac{SL^3}{EI} = \theta_n^4\frac{M}{m} + \frac{\theta_n^3(\cosh\theta_n \sin\theta_n + \sinh\theta_n \cos\theta_n)}{1 + \cos\theta_n \cosh\theta_n} \tag{A7}$$

This equation (or any of its previous forms) is the characteristic frequency equation of the beam-mass-spring system. A graphical representation of this equation is shown in figure 7, in which $\frac{SL^3}{EI}$ is plotted against θ for values of $\frac{M}{m} = 0, 2, 5, 10,$ and 50. The values of θ

corresponding to given values of SL^3/EI and M/m are the roots θ_n which characterize the modes of vibration.

Integration of equation (A5) with respect to time with the condition $(w)_{t=0} = 0$ gives for the deflection

$$w(x,\ t) = \frac{v}{c}\,\frac{L^2}{\rho}\sum_{n=1}^{\infty}\frac{\omega_0{}^2}{\omega_n{}^2}\,\frac{F\left(\theta_n\frac{x}{L}\right)}{\theta_n{}^2\,\Delta}\sin\omega_n t\ \mathit{1} \tag{A8}$$

From equation (A5) for velocity and equation (A8) for deflection the complete behavior of the idealized structure after landing may be found. The quantities of chief interest are the maximum bending stress, the maximum shear stress, the accelerations, and the force in the spring.

Maximum bending stresses.- The maximum bending stresses $\sigma(x,\ y,\ t)$, at any fiber distance y from the neutral axis, occur at the root and are given by the equation

$$\sigma(x,\ y,\ t) = Ey\left(\frac{\partial^2 w}{\partial x^2}\right)_{x=0}$$

$$= E\,\frac{v}{c}\,\frac{y}{\rho}\sum_{n=1}^{\infty}A_n\sin\omega_n t\ \mathit{1} \tag{A9}$$

where

$$A_n = 4\,\frac{\omega_0{}^2}{\omega_n{}^2}\,\frac{\sin\theta_n\sinh\theta_n\left(1+\cos\theta_n\cosh\theta_n\right)}{\left(1+3\dfrac{\omega_0{}^2}{\omega_n{}^2}\right)\left(1+\cos\theta_n\cosh\theta_n\right)^2+\dfrac{m}{M}\left(\cos\theta_n+\cosh\theta_n\right)^2}$$

Equation (A9) may be written in the form

$$\sigma = E\,\frac{v}{c}\,\frac{y}{\rho}\left(A_1\sin\omega_1 t + A_2\sin\omega_2 t + A_3\sin\omega_3 t + \ldots\right) \tag{A10}$$

In the expression in parentheses each term may be thought of as the contribution of a particular mode to the bending stress. The expression in parentheses may be denoted by $\bar{A}$ and equation (A10) becomes simply

$$\sigma = \bar{A} \frac{V}{c} \frac{y}{\rho} E$$

The values of A_1, A_2, and A_3 are plotted against ω_s/ω_c for $\frac{M}{m} = \frac{1}{2}$, 2, and 5 in figure 8, and the maximum value of $\bar{A}$ found to occur in the initial cycles of vibration by use of the first three terms of the series is plotted against ω_s/ω_c for $\frac{M}{m} = \frac{1}{2}$, 2, and 5, in figure 2. The quantity ω_s is the frequency of the system if the wing were rigid:

$$\omega_s = \sqrt{\frac{S}{M + m}} = \frac{\omega_0}{\sqrt{1 + \frac{m}{M}}}$$

The quantity ω_c is the frequency of the wing if the spring were infinitely stiff:

$$\omega_c = \rho c \frac{3.52}{L^2}$$

This equation represents the fundamental cantilever frequency. (See reference 1.) The ratio ω_s/ω_c is related to the parameters used in frequency equation (equation (A7)) by the relation

$$\frac{{\omega_s}^2}{{\omega_c}^2} = \frac{1}{12.4\left(1 + \frac{M}{m}\right)} \frac{SL^3}{EI}$$

<u>Maximum shear stress</u>.- The maximum average shear stress $\bar{\tau}(x, t)$ occurs at the root and is given by the equation

$$\overline{\tau}(x,\ t)_{x=0} = E\rho^2\left(\frac{\partial^3 w}{\partial x^3}\right)_{x=0}$$

$$= E\,\frac{V}{c}\,\frac{\rho}{L}\sum_{n=1}^{\infty} B_n \sin \omega_n t\ 1 \qquad \text{(A11)}$$

where

$$B_n = 4\,\frac{\omega_0^{\,2}}{\omega_n^{\,2}}\,\frac{\theta_n\left(\cosh\theta_n \sin\theta_n + \sinh\theta_n \cos\theta_n\right)\left(1 + \cos\theta_n \cosh\theta_n\right)}{\left(1 + 3\,\dfrac{\omega_0^{\,2}}{\omega_n^{\,2}}\right)\left(1 + \cos\theta_n \cosh\theta_n\right)^2 + \dfrac{m}{M}\left(\cos\theta_n + \cosh\theta_n\right)^2}$$

In figure 9 the values of B_1, B_2, and B_3 are plotted against ω_s/ω_c for $\frac{M}{m} = \frac{1}{2}$, 2, and 5. Equation (A11) may be written simply

$$\overline{\tau}(x,\ t)_{x=0} = \bar{\bar{B}}\,\frac{V}{c}\,\frac{\rho}{L}\,E$$

and in figure 10 the value of $\bar{\bar{B}}$ found by use of the first three modes with proper regard to phase is plotted against ω_s/ω_c for $\frac{M}{m} = \frac{1}{2}$, 2, and 5.

<u>Accelerations</u>.- The acceleration anywhere on the beam is found to be

$$\alpha(x,\ t) = \frac{\partial^2 w}{\partial t^2} = -\,\frac{V}{c}\,\frac{L^2}{\rho}\sum_{n=1}^{\infty} \omega_0^{\,2}\,\frac{F\left(\theta_n \frac{x}{L}\right)}{\theta_n^{\,2}\,\Delta} \sin \omega_n t\ 1 \qquad \text{(A12)}$$

Force in spring.- The force in the spring after landing is the spring stiffness times the displacement at the position $x=0$. By use of equation (A8), the force is found to be

$$F = Sw_{(x=0)} = S \frac{v}{c} \frac{L^2}{\rho} \sum_{n=1}^{\infty} \frac{\omega_0^{2}}{\omega_n^{2}} \frac{2\left(1 + \cos \theta_n \cosh \theta_n\right)}{\theta_n^{2} \Delta} \sin \omega_n t \quad 1 \qquad \text{(A13)}$$

APPENDIX B

APPROXIMATE SOLUTIONS

Stresses Computed for Rigid Wing

Method A - based on acceleration obtained with rigid wing (no structural elasticity).- If the wings of the structure shown in figure 1 were rigid the landing operation would be simply that of a rigid mass equal to $M + m$ alighting on a spring. The motion after arrest would be that of a simple oscillator having a mass $M + m$ and a maximum velocity v. The solution based on these assumptions is designated method A. The maximum acceleration for such an oscillator is

$$a = v\sqrt{\frac{S}{M + m}} = v\,\omega_s \tag{B1}$$

The bending stress is computed on the basis that the wing is loaded with a uniform load having an intensity equal to the mass per unit length times the maximum acceleration. From the static theory of the bending of a cantilever beam, the bending moment at the root resulting from this loading would be

$$M = a\,\frac{m}{L}\,\frac{L^2}{2} = \frac{amL}{2}$$

The bending stress due to this bending moment is

$$\sigma = \frac{My}{I} = \frac{amLy}{2I} = \frac{\omega_s\,mLvy}{2I} \tag{B2}$$

With the use of the notation $c^2 = \frac{Eg}{\gamma}$ and the equation for the cantilever frequency of a beam, $\omega_c = \rho c\,\frac{3.52}{L^2}$ (see reference 1), equation (B2) may be written

$$\sigma = 1.76\,\frac{\omega_s}{\omega_c}\,\frac{v}{c}\,\frac{y}{\rho}\,E$$

$$= \bar{A}\,\frac{v}{c}\,\frac{y}{\rho}\,E \tag{B3}$$

This equation is of the same form as equation (A9). It is noted that the stress varies linearly with ω_s/ω_c, and has no explicit dependence on the ratio M/m. The value of $\bar{A}$ obtained by method A is plotted against ω_s/ω_c in figure 3.

Method B - based on reaction obtained with elastic wing.- In method B, a force equal to the maximum force given by equation (A13) is applied at the root. The rigid wing and fuselage mass then have an acceleration

$$a = \frac{F_{max}}{M + m}$$

With this acceleration applied to the root, the static stresses induced at the root would be

$$\begin{aligned}
\sigma &= \frac{My}{I} \\
&= \frac{amLy}{2I} \\
&= \frac{v}{c}\frac{y}{\rho}\frac{mL^3}{2I}\frac{S}{M + m}\left[\sum_{n=1}^{\infty}\frac{\omega_0^{\,2}}{\omega_n^{\,2}}\frac{2(1 + \cos\theta_n \cosh\theta_n)}{\theta_n^{\,2}\,\Delta}\sin\omega_n t\ 1\right]_{max} \\
&= E\frac{v}{c}\frac{y}{\rho}\left[\sum_{n=1}^{\infty}\frac{\omega_0^{\,2}\omega_s^{\,2}}{\omega_n^{\,4}}\frac{\theta_n^{\,2}(1 + \cos\theta_n \cosh\theta_n)}{\Delta}\sin\omega_n t\ 1\right]_{max} \\
&= E\frac{v}{c}\frac{y}{\rho}\bar{A} \qquad \text{(B4)}
\end{aligned}$$

The value of $\bar{A}$ obtained by method B is plotted against ω_s/ω_c for for $\frac{M}{m} = 2$ in figure 3.

Method C - based on acceleration obtained with elastic wing.- In method C, the root of the rigid wing is given an acceleration equal to the maximum acceleration given by equation (A12). The bending stresses at the root is then

$$\sigma = \frac{My}{I}$$

$$= \frac{amLy}{2I}$$

$$= \frac{v}{c}\,\frac{y}{\rho}\,\frac{mL^3}{2I}\left[\sum_{n=1}^{\infty} \omega_0^{\,2}\,\frac{F(0)}{\theta_n^{\,2}\,\Delta}\,\sin\,\omega_n t\;\mathit{1}\right]_{max}$$

$$= E\,\frac{v}{c}\,\frac{y}{\rho}\left[\sum_{n=1}^{\infty} \frac{\omega_0^{\,2}}{\omega_n^{\,2}}\,\frac{\theta_n^{\,2}(1 + \cos\,\theta_n\,\cosh\,\theta_n)}{\Delta}\,\sin\,\omega_n t\;\mathit{1}\right]_{max}$$

$$= E\,\frac{v}{c}\,\frac{y}{\rho}\,\bar{A} \tag{B5}$$

The value of $\bar{A}$ obtained by method C is plotted against ω_s/ω_c for $\frac{M}{m} = 2$ in figure 3.

Stresses Computed for Elastic Wing

<u>Method D - based on reaction obtained with rigid wing</u>.- The reaction that results from landing when the wing is rigid will vary sinusoidally with an amplitude $S\frac{v}{\omega_s}$, thus

$$R = S\,\frac{v}{\omega_s}\,\sin\,\omega_s t$$

If, in accordance with method D, this reaction were suddenly applied to the root of the elastic wing, bending vibrations would be set up in the wing. The response can be found in a manner similar to that used in the exact solution of appendix A. The only difference is that the last boundary condition is changed. The last boundary condition for this case is

$$M\left(\frac{\partial^2 w}{\partial t^2}\right)_{x=0} + EI\left(\frac{\partial^3 w}{\partial x^3}\right)_{x=0} = -S\,\frac{v}{\omega_s}\sin \omega_s t \; 1$$

With this and the remaining boundary conditions the deflections and bending stresses are then found as in the case of the exact solution. The maximum bending stress for this case is found to be

$$\sigma = E\,\frac{v}{c}\,\frac{y}{\rho}\left[\sum_{n=0}^{\infty} E_n \sin \omega_n t \; 1\right]_{max} \tag{B6}$$

where

$$E_n = \frac{4\,\dfrac{\omega_0^{\,2}}{\omega_n^{\,2}}\,\dfrac{M}{m}\sin\theta_n \sinh\theta_n}{\left(1 - \dfrac{\omega_s^{\,2}}{\omega_n^{\,2}}\right)\left\{\dfrac{M}{m}\left[\left(1 + \cos\theta_n \cosh\theta_n\right) + \theta_n\left(\sinh\theta_n \cos\theta_n - \cosh\theta_n \sin\theta_n\right)\right] + 2\cos\theta_n \cosh\theta_n\right\}}$$

In this expression, θ_n is the nth positive root of the equation

$$1 + \cos\theta_n \cosh\theta_n + \frac{M}{m\theta_n}\left(\cosh\theta_n \sin\theta_n + \sinh\theta_n \cos\theta_n\right) = 0$$

If $\bar{A}$ is used to replace the bracketed term in equation (B6)

$$\sigma = \bar{A} \frac{v}{c} \frac{y}{\rho} E$$

The value of $\bar{A}$ obtained by method D is plotted against ω_s/ω_c for $\frac{M}{m} = 2$ in figure 6.

Method E - based on acceleration obtained with rigid wing.- The accelerations that result from landing when the wing is rigid will vary sinusoidally with an amplitude given by equation (B1), thus

$$a = v\ \omega_s \sin \omega_s t$$

If, in accordance with method E, the root of the elastic wing were suddenly given an acceleration characterized by the equation, the wing would be set into bending oscillations. Again, the oscillations or response can be found in a manner similar to that used in computing the landing response by the exact solution given in appendix A. The last boundary condition for this case is, however,

$$\left(\frac{\partial^2 w}{\partial t^2}\right)_{x=0} = v\ \omega_s \sin \omega_s t \ \mathit{1}$$

With this and the remaining boundary conditions, the deflections and bending stresses are then found as in the case of the exact solution. The maximum root bending stress is found to be

$$\sigma = E \frac{v}{c} \frac{y}{\rho} \left[F_0 \sin \omega_s t \ \mathit{1} + \sum_{n=1}^{\infty} \frac{4\frac{\omega_s^2}{\omega_n^2}}{\frac{\omega_s^2}{\omega_n^2} - 1} F_n \sin \omega_n t \ \mathit{1} \right]_{max} \tag{B7}$$

where

$$F_0 = \frac{\sin\theta_0 \sinh\theta_0}{1 + \cos\theta_0 \cosh\theta_0}$$

$$F_n = \frac{\sin\theta_n \sinh\theta_n}{\theta_n\left(\sin\theta_n \cosh\theta_n - \cos\theta_n \sinh\theta_n\right)}$$

$$\theta_0 = \sqrt{3.52\,\frac{\omega_s^2}{\omega_c^2}}$$

$$\theta_n = \text{nth positive root of } 1 + \cos\theta_n \cosh\theta_n = 0$$

$$\omega_s = \sqrt{\frac{S}{M + m}}$$

$$\omega_n = \rho c\,\frac{\theta_n^2}{L^2}$$

If $\bar{A}$ is used to replace the bracketed term in equation (B7)

$$\sigma = \bar{A}\,\frac{v}{c}\,\frac{y}{\rho}\,E$$

The value of $\bar{A}$ obtained by method E is plotted against ω_s/ω_c for $\frac{M}{m} = 2$ in figure 6.

Method F - Biot and Bisplinghoff method.- In the statistical approach suggested by Biot and Bisplinghoff (reference 2), the maximum force and duration of impact have to be known. This method (designated method F herein) was applied to the case considered by taking the maximum force equal to the maximum value given by the exact solution (equation (A13)) and by taking the vertical impulse period T_I equal to one-half the natural period of the airplane structure with rigid wings and with the bottom of the spring fixed in position; thus $T_I = \frac{1}{2} T_S$, where $T_S = \frac{2\pi}{\omega_s}$.

The ratio of the impulse period to the period of the nth mode $\frac{T_I}{T_n}$ is found by the following consideration. The natural frequency of the free-free modes of the structure, which are the modes used in the Biot and Bisplinghoff method, are found from the equation for frequency given in the exact solution (appendix A):

$$\omega_n = \rho c \frac{{\theta_n}^2}{L^2}$$

where θ_n is taken to correspond to a structure without landing springs and may be taken from figure 7 at $\frac{SL^3}{EI} = 0$. With the use of the equation for frequency of a cantilever, $\omega_c = \rho c \frac{3.52}{L^2}$, ω_n may be written

$$\omega_n = \omega_c \frac{{\theta_n}^2}{3.52}$$

Division through by ω_s and use of the relation between period and frequency result in the relations

$$\frac{\omega_n}{\omega_s} = \frac{\omega_c}{\omega_s} \frac{{\theta_n}^2}{3.52} = \frac{T_S}{T_n}$$

Since T_S is taken as $2T_I$, the equation for $\frac{T_I}{T_n}$ may be written directly

$$\frac{T_I}{T_n} = \frac{1}{2} \frac{1}{\omega_s/\omega_c} \frac{{\theta_n}^2}{3.52}$$

For the ratio $\frac{M}{m} = 2$, the value of ${\theta_1}^2$ is 4.00 and ${\theta_2}^2$ is 23.00. The value of ${\theta_3}^2$ is not given because it was found that the third mode could be neglected.

In computing stresses, the response factor was taken directly from the envelope curve given in figure 13 of reference 4. The values of stress coefficient obtained by method F are shown in figure 6.

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2. Biot, M. A., and Bisplinghoff, R. L.: Dynamic Loads on Airplane Structures During Landing. NACA ARR No. 4H10, 1944.

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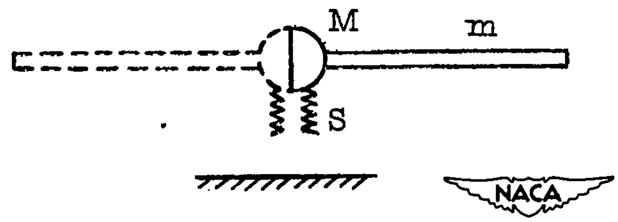

Figure 1.- Simplified structure used in landing analysis.

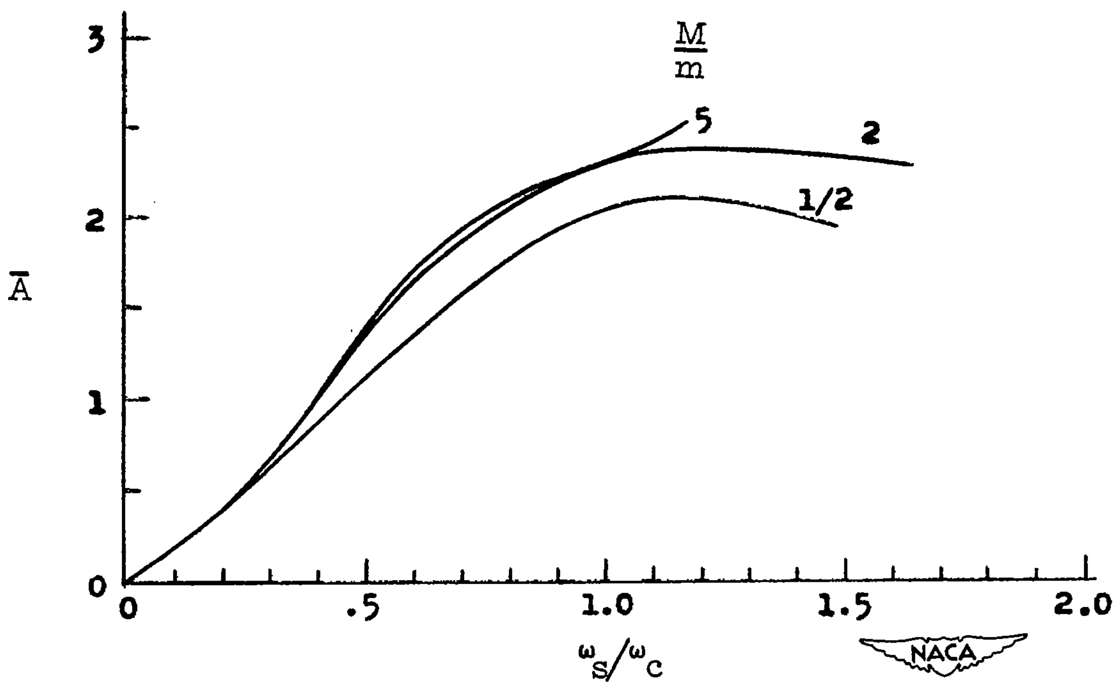

Figure 2.- Exact solution for maximum root bending-stress coefficient. $\sigma = \bar{A}\frac{V}{c}\frac{y}{\rho}E$.

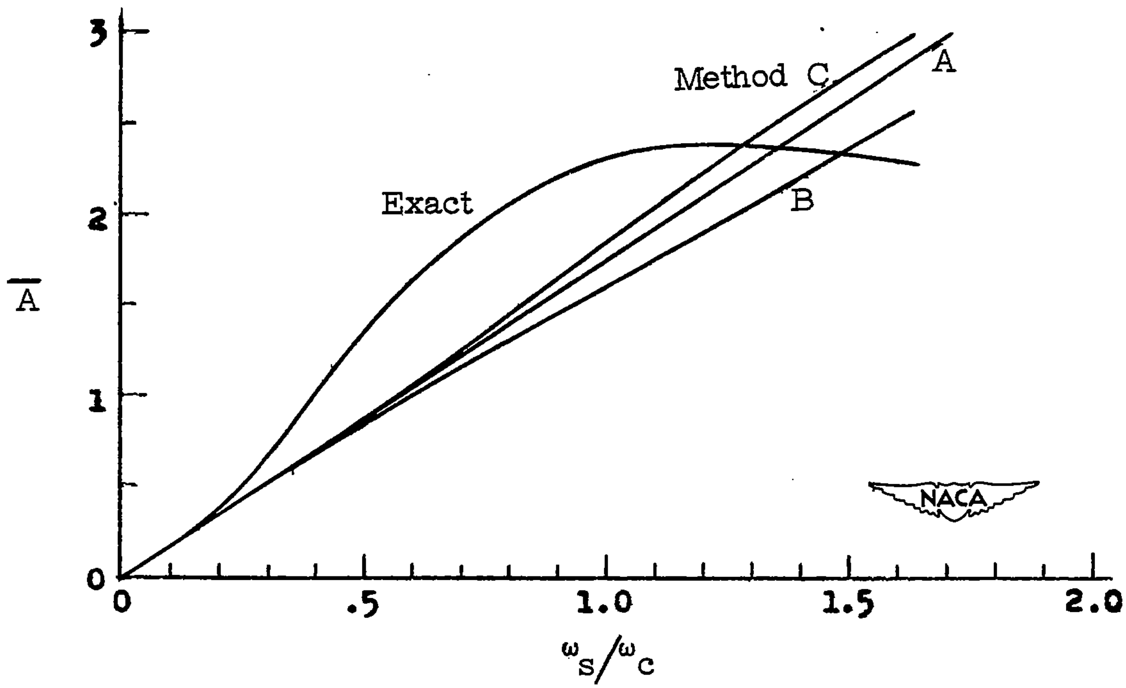

Figure 3.- Bending-stress coefficients when structural elasticity is neglected in computing stresses (that is, in stage 2). $\frac{M}{m} = 2$. $\sigma = \overline{A} \frac{v}{c} \frac{y}{\rho} E$.

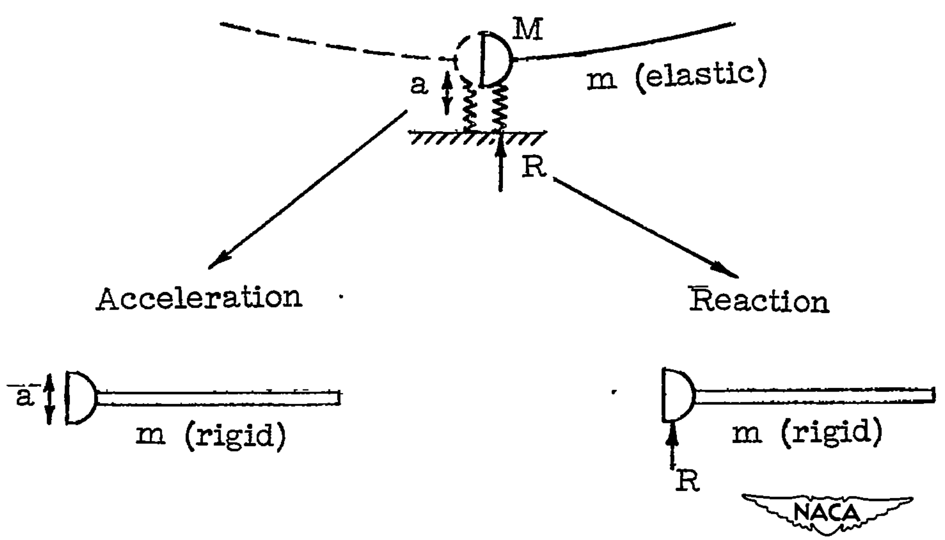

Figure 4.- Physical assumptions when structural elasticity is considered in determining reaction or acceleration but neglected in computing stresses.

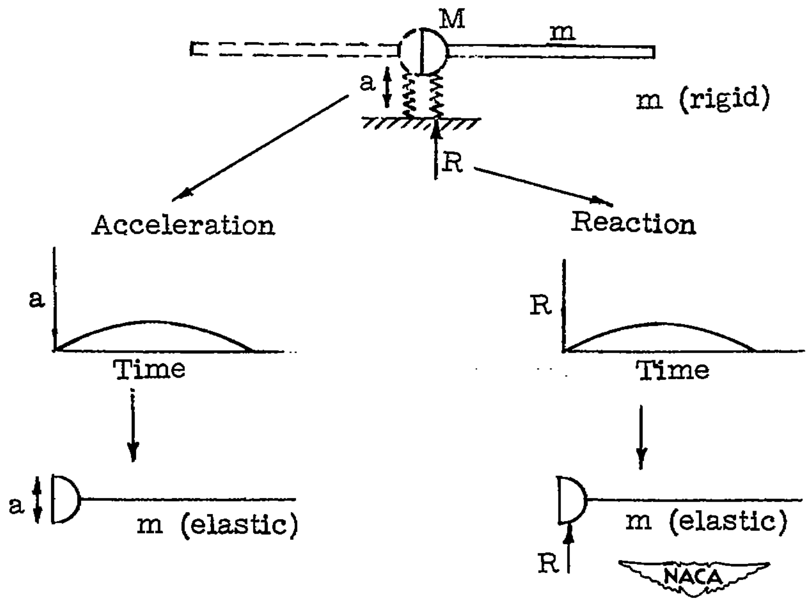

Figure 5.- Physical assumptions when structural elasticity is neglected in determining reaction or acceleration but is considered in computing stresses.

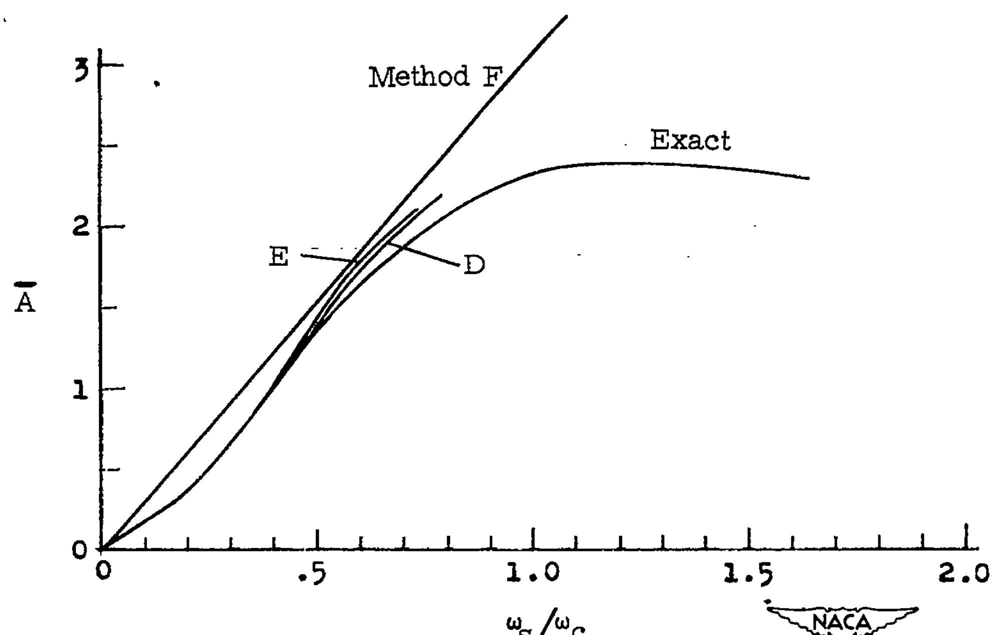

Figure 6.- Bending-stress coefficients when structural elasticity is neglected in computing reaction or acceleration (that is, in stage 1). $\frac{M}{m} = 2$. $\sigma = \bar{A}\frac{v}{c}\frac{y}{\rho}E$.

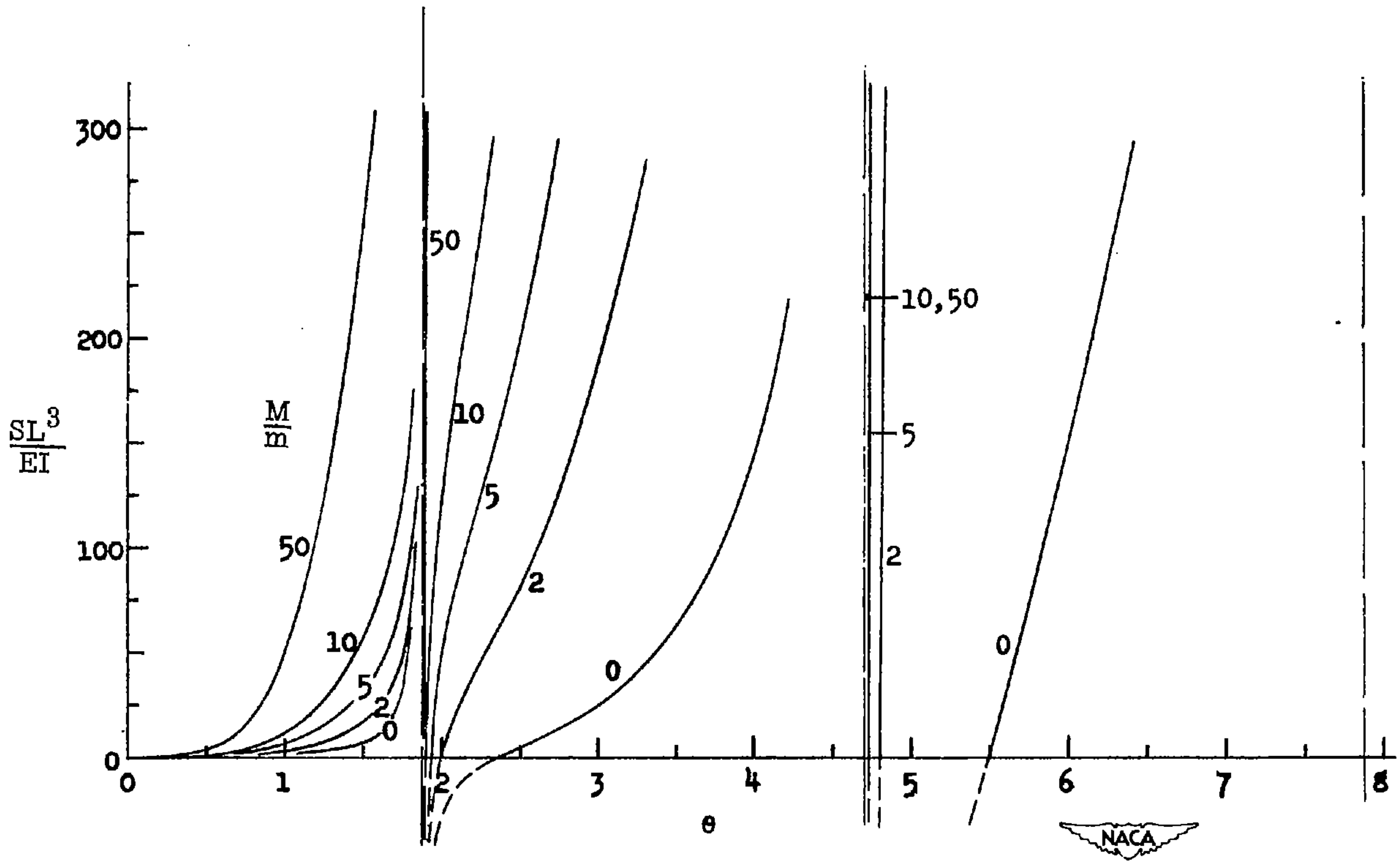

Figure 7.- Graph of characteristic frequency equation for several values of ratio of fuselage mass to wing mass.

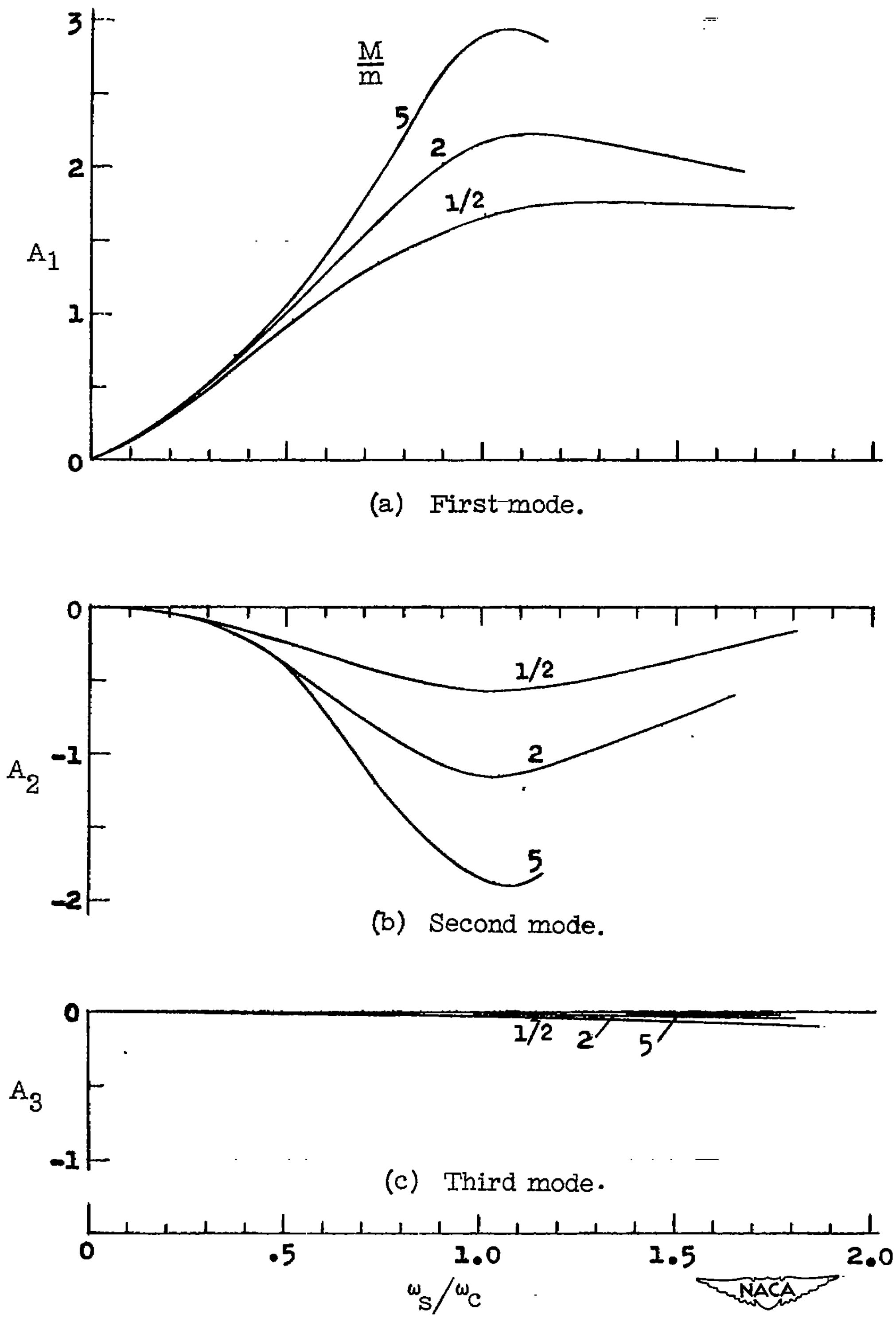

Figure 8.- Bending-stress coefficient at root for first three modes.

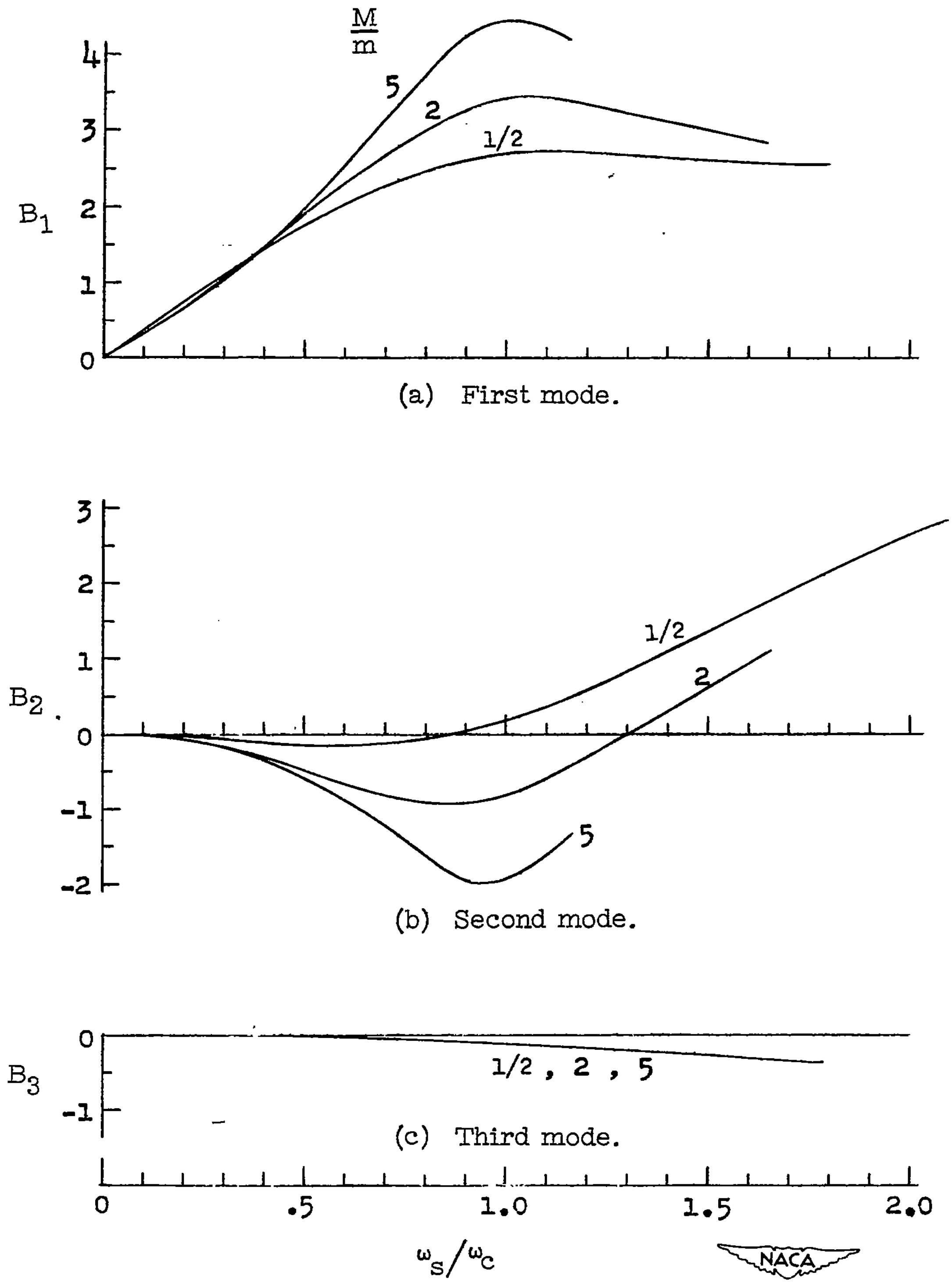

Figure 9.- Shear-stress coefficient at root for first three modes.

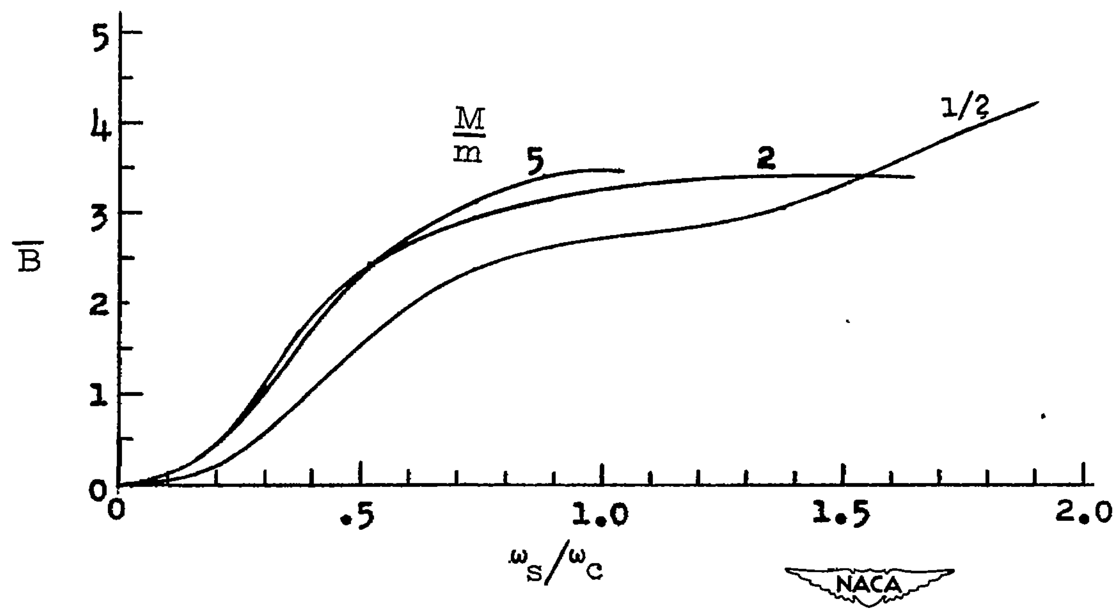

Figure 10.- Shear-stress coefficient at root for maximum average shear stress. $\overline{\tau} = \overline{B}\frac{V}{c}\frac{\rho}{L} E$.